

# UNIQUE EXPANSIONS AND INTERSECTIONS OF CANTOR SETS

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ABSTRACT. To each  $\alpha \in (1/3, 1/2)$  we associate the Cantor set

$$\Gamma_\alpha := \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{0, 1\}, i \geq 1 \right\}.$$

In this paper we consider the intersection  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  for any translation  $t \in \mathbb{R}$ . We pay special attention to those  $t$  with a unique  $\{-1, 0, 1\}$   $\alpha$ -expansion, and study the set

$$D_\alpha := \{\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \text{ has a unique } \{-1, 0, 1\} \alpha\text{-expansion}\}.$$

We prove that there exists a transcendental number  $\alpha_{KL} \approx 0.39433\dots$  such that:  $D_\alpha$  is finite for  $\alpha \in (\alpha_{KL}, 1/2)$ ,  $D_{\alpha_{KL}}$  is infinitely countable, and  $D_\alpha$  contains an interval for  $\alpha \in (1/3, \alpha_{KL})$ . We also prove that  $D_\alpha$  equals  $[0, \frac{\log 2}{-\log \alpha}]$  if and only if  $\alpha \in (1/3, \frac{3-\sqrt{5}}{2}]$ .

As a consequence of our investigation we prove some results on the possible values of  $\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t))$  when  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  is a self-similar set. We also give examples of  $t$  with a continuum of  $\{-1, 0, 1\}$   $\alpha$ -expansions for which we can explicitly calculate  $\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t))$ , and for which  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  is a self-similar set. We also construct  $\alpha$  and  $t$  for which  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  contains only transcendental numbers.

Our approach makes use of digit frequency arguments and a lexicographic characterisation of those  $t$  with a unique  $\{-1, 0, 1\}$   $\alpha$ -expansion.

## 1. INTRODUCTION

To each  $\alpha \in (0, 1/2)$  we associate the contracting similarities  $f_0(x) = \alpha x$  and  $f_1(x) = \alpha(x+1)$ . The middle  $(1-2\alpha)$  Cantor set  $\Gamma_\alpha$  is defined to be the unique compact non-empty set satisfying the equation

$$\Gamma_\alpha = f_0(\Gamma_\alpha) + f_1(\Gamma_\alpha).$$

It is easy to see that the maps  $\{f_0, f_1\}$  satisfy the strong separation condition. Thus  $\dim_H(\Gamma_\alpha) = \dim_B(\Gamma_\alpha) = \frac{\log 2}{-\log \alpha}$ , where  $\dim_H$  and  $\dim_B$  denote the Hausdorff dimension and box dimension respectively.

A natural and well studied question is “What are the properties of the intersection  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ ?” This question has been studied by many authors. We refer the reader

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to [10, 16, 18, 17, 15] and the references therein for more information. As we now go on to explain, when  $\alpha \in (0, 1/3]$  the set  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  is well understood, however when  $\alpha \in (1/3, 1/2)$  additional difficulties arise.

Note that  $\Gamma_\alpha \cap (\Gamma_\alpha + t) \neq \emptyset$  if and only if  $t \in \Gamma_\alpha - \Gamma_\alpha$ . Thus it is natural to investigate the difference set  $\Gamma_\alpha - \Gamma_\alpha$ , which is the self-similar set generated by the iterated function system  $\{f_{-1}, f_0, f_1\}$ , where  $f_{-1}(x) = \alpha(x - 1)$ . Alternatively, one can write

$$\Gamma_\alpha - \Gamma_\alpha := \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{-1, 0, 1\}, i \geq 1 \right\}.$$

Importantly, for  $\alpha \in (0, 1/3)$  each  $t \in \Gamma_\alpha - \Gamma_\alpha$  has a unique  $\alpha$ -expansion with alphabet  $\{-1, 0, 1\}$ , i.e., there exists a unique sequence  $(t_i) \in \{-1, 0, 1\}^{\mathbb{N}}$  such that  $t = \sum t_i \alpha^i$ . When  $\alpha = 1/3$  there is a countable set of  $t$  with precisely two  $\alpha$ -expansions. These  $t$  are well understood and do not pose any real difficulties, thus in what follows we suppress the case where  $t$  has two  $\alpha$ -expansions.

For  $\alpha \in (0, 1/3]$  let  $t \in \Gamma_\alpha - \Gamma_\alpha$  have a unique  $\alpha$ -expansion  $(t_i)$ . Then the sequence  $(t_i)$  provides a useful description of the set  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ . Indeed, we can write (cf. [18])

$$(1.1) \quad \Gamma_\alpha \cap (\Gamma_\alpha + t) = \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{0, 1\} \cap (\{0, 1\} + t_i) \right\}.$$

With this new interpretation many questions regarding the set  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  can be re-interpreted and successfully answered using combinatorial properties of the  $\alpha$ -expansion  $(t_i)$ .

The straightforward description of  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  provided by (1.1) does not exist for  $\alpha \in (1/3, 1/2)$  and a generic  $t \in \Gamma_\alpha - \Gamma_\alpha$ . The set  $\Gamma_\alpha - \Gamma_\alpha$  is still a self-similar set generated by the transformations  $\{f_{-1}, f_0, f_1\}$ , however this set is now equal to the interval  $[\frac{-\alpha}{1-\alpha}, \frac{\alpha}{1-\alpha}]$  and the good separation properties that were present in the case where  $\alpha \in (0, 1/3]$  no longer exist. It is possible that a  $t \in \Gamma_\alpha - \Gamma_\alpha$  could have many  $\alpha$ -expansions. In fact it can be shown that Lebesgue almost every  $t \in \Gamma_\alpha - \Gamma_\alpha$  has a continuum of  $\alpha$ -expansions (cf. [4, 21, 22]). Thus within the parameter space  $(1/3, 1/2)$  we are forced to have the following more complicated interpretation of  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  (cf. [18, Lemma 3.3])

$$(1.2) \quad \Gamma_\alpha \cap (\Gamma_\alpha + t) = \bigcup_{\tilde{t}} \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{0, 1\} \cap (\{0, 1\} + \tilde{t}_i) \right\},$$

where the union is over all  $\alpha$ -expansions  $\tilde{t} = (\tilde{t}_i)$  of  $t$ . As stated above, for a generic  $t$  this union is uncountable, this makes many questions regarding the set  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  intractable. In what follows we focus on the case where  $t$  has a unique  $\alpha$ -expansion. For these  $t$  the description of  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  given by (1.2) simplifies to that given by (1.1).

We now introduce some notation. For  $\alpha \in (0, 1/2)$  let

$$\mathcal{U}_\alpha := \left\{ t \in \Gamma_\alpha - \Gamma_\alpha : t \text{ has a unique } \alpha\text{-expansion w.r.t. the alphabet } \{-1, 0, 1\} \right\}.$$

Within this paper one of our main objects of study is the following set

$$D_\alpha := \left\{ \dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \in \mathcal{U}_\alpha \right\}.$$

In particular we will prove the following theorems.

**Theorem 1.1.** *There exists a transcendental number  $\alpha_{KL} \approx 0.39433 \dots$  such that:*

(1) *For  $\alpha \in (\alpha_{KL}, 1/2)$  there exists  $n^* \in \mathbb{N}$  such that*

$$D_\alpha = \left\{ 0, \frac{\log 2}{-\log \alpha} \right\} \cup \left\{ \frac{\log 2}{\log \alpha} \sum_{i=1}^n \left( \frac{-1}{2} \right)^i : 1 \leq n \leq n^* \right\}.$$

(2)

$$D_{\alpha_{KL}} = \left\{ 0, \frac{\log 2}{-\log \alpha_{KL}}, \frac{\log 2}{-3 \log \alpha_{KL}} \right\} \cup \left\{ \frac{\log 2}{\log \alpha_{KL}} \sum_{i=1}^n \left( \frac{-1}{2} \right)^i : 1 \leq n < \infty \right\}.$$

(3)  *$D_\alpha$  contains an interval if  $\alpha \in (1/3, \alpha_{KL})$ .*

In [18] it was asked “When  $\alpha \in (1/3, 1/2)$  what are the possible values of  $\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t))$  for  $t \in \Gamma_\alpha - \Gamma_\alpha$ ?” The following theorem provides a partial solution to this problem.

**Theorem 1.2.** (1) *If  $\alpha \in (1/3, \frac{3-\sqrt{5}}{2}]$  then  $D_\alpha = [0, \frac{\log 2}{-\log \alpha}]$ .*

(2) *If  $\alpha \in (\frac{3-\sqrt{5}}{2}, 1/2)$  then  $D_\alpha$  is a proper subset of  $[0, \frac{\log 2}{-\log \alpha}]$ .*

Amongst  $\Gamma_\alpha - \Gamma_\alpha$  a special class of  $t$  are those for which  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  is a self-similar set. Determining whether  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  is a self-similar set is a difficult problem for a generic  $t$  with many  $\alpha$ -expansions, thus we consider only those  $t \in \mathcal{U}_\alpha$ . Let

$$S_\alpha := \left\{ t \in \mathcal{U}_\alpha : \Gamma_\alpha \cap (\Gamma_\alpha + t) \text{ is a self-similar set} \right\}.$$

We prove the following result.

**Theorem 1.3.** (1) *If  $\alpha \in (1/3, \frac{3-\sqrt{5}}{2}]$  then  $\{\dim(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \in S_\alpha\}$  is dense in  $[0, \frac{\log 2}{-\log \alpha}]$ .*

(2) *If  $\alpha \in (\frac{3-\sqrt{5}}{2}, 1/2)$  then  $\{\dim(\Gamma_\alpha \cap (\Gamma_\alpha + t)) : t \in S_\alpha\}$  is not dense in  $[0, \frac{\log 2}{-\log \alpha}]$ .*

What remains of this paper is arranged as follows. In Section 2 we recall the necessary preliminaries from expansions in non-integer bases, and recall an important result of [18] that connects the dimension of  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  with the frequency of 0's in the  $\alpha$ -expansion  $(t_i)$ . In Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2 and Theorem 1.3. In Section 5 we include some examples. We give two examples of an  $\alpha \in (1/3, 1/2)$ , and  $t \in \Gamma_\alpha - \Gamma_\alpha$  with a continuum of  $\alpha$ -expansions, for which we can explicitly calculate  $\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t))$ . The techniques used in our first example can be applied to the more general case where  $\alpha$  is the reciprocal of a Pisot number and  $t \in \mathbb{Q}(\alpha)$ . Our second example demonstrates that it is possible for  $t$  to have a continuum of  $\alpha$ -expansions and for  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  to be a self-similar set. Moreover, both of these examples show that it is possible to have

$$\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) > \sup_{\tilde{t}} \dim_H \left( \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{0, 1\} \cap (\{0, 1\} + \tilde{t}_i) \right\} \right).$$

Our final example demonstrates the existence of  $\alpha \in (1/3, 1/2)$  and  $t \in \Gamma_\alpha - \Gamma_\alpha$  for which  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  contains only transcendental numbers.

## 2. PRELIMINARIES

Let  $M \in \mathbb{N}$  and  $\alpha \in [\frac{1}{M+1}, 1)$ . Given  $x \in I_{\alpha, M} := [0, \frac{M\alpha}{1-\alpha}]$  we call a sequence  $(\epsilon_i) \in \{0, \dots, M\}^{\mathbb{N}}$  an  $\alpha$ -*expansion* for  $x$  with alphabet  $\{0, \dots, M\}$  if

$$x = \sum_{i=1}^{\infty} \epsilon_i \alpha^i.$$

This method of representing real numbers was pioneered in the early 1960's in the papers of Rényi [20] and Parry [19]. One aspect of these representations that makes them interesting is that for  $\alpha \in (\frac{1}{M+1}, 1)$  a generic  $x \in I_{\alpha, M}$  has many  $\alpha$ -expansions (cf. [4, 21, 22]). This naturally leads researchers to study the set of  $x \in I_{\alpha, M}$  with a unique  $\alpha$ -expansion, the so called *univoque set*. We define this set as follows

$$\mathcal{U}_{\alpha, M} := \left\{ x \in I_{\alpha, M} : x \text{ has a unique } \alpha\text{-expansion w.r.t. the alphabet } \{0, 1, \dots, M\} \right\}.$$

Accordingly, let  $\tilde{\mathcal{U}}_\alpha$  denote the set of corresponding expansions, i.e.,

$$\tilde{\mathcal{U}}_{\alpha, M} := \left\{ (\epsilon_i) \in \{0, \dots, M\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \epsilon_i \alpha^i \in \mathcal{U}_{\alpha, M} \right\}.$$

The sets  $\mathcal{U}_{\alpha, M}$  and  $\tilde{\mathcal{U}}_{\alpha, M}$  have been studied by many authors. For more information on these sets we refer the reader to [7, 5, 9, 6, 11, 12] and the references therein. Before continuing with our discussion of the sets  $\mathcal{U}_{\alpha, M}$  and  $\tilde{\mathcal{U}}_{\alpha, M}$  we make a brief remark. In

the introduction we were concerned with  $\alpha$ -expansions with digit set  $\{-1, 0, 1\}$ , not with a digit set  $\{0, \dots, M\}$ . However, all of the result that are stated below for a digit set  $\{0, \dots, M\}$  also hold for any digit set of  $M + 1$  consecutive integers  $\{s, \dots, s + M\}$ . In particular, statements that are true for the digit set  $\{0, 1, 2\}$  translate to results for the digit set  $\{-1, 0, 1\}$  by performing the substitutions  $0 \rightarrow -1, 1 \rightarrow 0, 2 \rightarrow 1$ .

We now define the lexicographic order and introduce some notations. Given two finite sequences  $\omega = (\omega_1, \dots, \omega_n), \omega' = (\omega'_1, \dots, \omega'_n) \in \{0, \dots, M\}^n$ , we say that  $\omega$  is less than  $\omega'$  with respect to the lexicographic order, or simply write  $\omega \prec \omega'$ , if  $\omega_1 < \omega'_1$  or if there exists  $1 \leq j < n$  such that  $\omega_i = \omega'_i$  for  $1 \leq i \leq j$  and  $\omega_{j+1} < \omega'_{j+1}$ . One can also define the relations  $\preceq, \succ, \succeq$  in the natural way, and we can extend the lexicographic order to infinite sequences. We define the *reflection* of a finite/infinite sequence  $(\epsilon_i)$  to be  $(\overline{\epsilon_i}) = (M - \epsilon_i)$ , where the underlying  $M$  should be obvious from our context. For a finite sequence  $\omega = (\omega_1, \dots, \omega_n)$  we define the finite sequence  $\omega^-$  to be  $(\omega_1, \dots, \omega_n - 1)$ . Moreover, we denote the concatenation of  $\omega$  with itself  $n$  times by  $\omega^n$ , we also let  $\omega^\infty$  denote the infinite sequence obtained by indefinitely concatenating  $\omega$  with itself.

Given  $x \in I_{\alpha, M}$  we define the *greedy*  $\alpha$ -expansion of  $x$  to be the lexicographically largest sequence amongst the  $\alpha$ -expansions of  $x$ . We define the *quasi-greedy*  $\alpha$ -expansion of  $x$  to be the lexicographically largest infinite sequence amongst the  $\alpha$ -expansions of  $x$ . Here we call a sequence  $(\epsilon_i)$  *infinite* if  $\epsilon_i \neq 0$  for infinitely many  $i$ . When studying the sets  $\mathcal{U}_{\alpha, M}$  and  $\tilde{\mathcal{U}}_{\alpha, M}$  a pivotal role is played by the quasi-greedy  $\alpha$ -expansion of 1. In what follows we will denote the quasi-greedy  $\alpha$ -expansion of 1 by  $(\delta_i(\alpha))$ . The importance of the sequence  $(\delta_i(\alpha))$  is well demonstrated by the following technical lemma proved in [19] (see also, [7, 6]).

**Lemma 2.1.** *A sequence  $(\epsilon_i)$  belongs to  $\tilde{\mathcal{U}}_{\alpha, M}$  if and only if the following two conditions are satisfied:*

$$\begin{aligned} (\epsilon_{n+i}) &\prec (\delta_i(\alpha)) \text{ whenever } \epsilon_1 \dots \epsilon_n \neq M^n \\ (\overline{\epsilon_{n+i}}) &\prec (\delta_i(\alpha)) \text{ whenever } \epsilon_1 \dots \epsilon_n \neq 0^n \end{aligned}$$

Lemma 2.1 provides a useful characterisation of the set  $\tilde{\mathcal{U}}_{\alpha, M}$  in terms of the sequence  $(\delta_i(\alpha))$ . The following lemma describes the sequences  $(\delta_i(\alpha))$ .

**Lemma 2.2.** *Let  $M \in \mathbb{N}$ ,  $\alpha \in [\frac{1}{M+1}, 1)$  and  $(\delta_i(\alpha))$  be the quasi-greedy  $\alpha$ -expansion of 1. The map  $\alpha \rightarrow (\delta_i(\alpha))$  is a strictly decreasing bijection from the interval  $[\frac{1}{M+1}, 1)$  onto the set of all infinite sequences  $(\delta_i) \in \{0, \dots, M\}^\mathbb{N}$  satisfying*

$$\delta_{k+1}\delta_{k+2} \dots \preceq \delta_1\delta_2 \dots \text{ for all } k \geq 0.$$

The following technical result was proved in [18, Theorem 3.4] for  $\alpha \in (0, 1/3]$ , where importantly every  $t$  has a unique  $\alpha$ -expansion, except for  $\alpha = 1/3$  where countably many  $t$  have two  $\alpha$ -expansions. The proof translates over to the more general case where  $\alpha \in (1/3, 1/2)$  and  $t \in \mathcal{U}_\alpha$ .

**Lemma 2.3.** *Let  $\alpha \in (1/3, 1/2)$  and  $t \in \mathcal{U}_\alpha$ , then*

$$\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) = \frac{\log 2}{-\log \alpha} \underline{d}((t_i)),$$

where

$$\underline{d}((t_i)) := \liminf_{n \rightarrow \infty} \frac{\#\{1 \leq i \leq n : t_i = 0\}}{n}.$$

Lemma 2.3 will be a vital tool in proving Theorems 1.1 and 1.2. This result allows us to reinterpret Theorems 1.1 and 1.2 in terms of statements regarding the frequency of 0's that can occur within an element of  $\tilde{\mathcal{U}}_\alpha$ .

In what follows, for an infinite sequence  $(t_i) \in \{-1, 0, 1\}^\mathbb{N}$  we will use the notation

$$\bar{d}((t_i)) := \limsup_{n \rightarrow \infty} \frac{\#\{1 \leq i \leq n : t_i = 0\}}{n}.$$

When this limit exists, i.e.,  $\underline{d}((t_i)) = \bar{d}((t_i))$ , we simply use  $d((t_i))$ . For a word  $t_1 \dots t_n \in \{-1, 0, 1\}^n$  we will use the notation

$$d(t_1 \dots t_n) := \frac{\#\{1 \leq i \leq n : t_i = 0\}}{n}.$$

### 3. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. We start by defining the Thue-Morse sequence and its natural generalisation.

Let  $(\tau_i)_{i=0}^\infty \in \{0, 1\}^\mathbb{N}$  denote the classical Thue-Morse sequence. This sequence is defined iteratively as follows. Let  $\tau_0 = 0$  and if  $\tau_i$  is defined for some  $i \geq 0$ , set  $\tau_{2i} = \tau_i$  and  $\tau_{2i+1} = 1 - \tau_i$ . Then the sequence  $(\tau_i)_{i=0}^\infty$  begins with

$$0110 \ 1001 \ 1001 \ 0110 \ 1001 \ 01100110 \dots$$

For more on this sequence we refer the reader to [1]. Within expansions in non-integer bases the sequence  $(\tau_i)_{i=0}^\infty$  is important for many reasons. In [13] Komornik and Loreti proved that the unique  $\alpha$  for which  $(\delta_i(\alpha)) = (\tau_i)_{i=1}^\infty$  is the largest  $\alpha \in (1/2, 1)$  for which 1 has a unique  $\alpha$ -expansion. This  $\alpha$  has since become known as the *Komornik-Loreti constant*. Interesting connections between the size of  $\mathcal{U}_\alpha$  and the Komornik Loreti constant were

made in [9]. Using the Thue-Morse sequence we define a new sequence  $(\lambda_i) \in \{-1, 0, 1\}^{\mathbb{N}}$  as follows

$$(\lambda_i)_{i=1}^{\infty} = (\tau_i - \tau_{i-1})_{i=1}^{\infty}.$$

We denote the unique  $\alpha \in (1/2, 1)$  for which  $\sum_{i=1}^{\infty} (1 + \lambda_i) \alpha^i = 1$  by  $\alpha_{KL}$ . Our choice of subscript is because the constant  $\alpha_{KL}$  is a type of generalised Komornik-Loreti constant. This number is transcendental (cf. [14]) and is approximately 0.39433. This sequence satisfies the property

$$(3.1) \quad \begin{aligned} \lambda_1 &= 1, & \lambda_{2^{n+1}} &= 1 - \lambda_{2^n}; \\ \lambda_{2^n+i} &= -\lambda_i & \text{for any } 1 \leq i < 2^n. \end{aligned}$$

This property can be deduced directly from [14, Lemma 5.2]. So, the sequence  $(\lambda_i)_{i=1}^{\infty}$  starts at

$$10(-1)1(-1)010(-1)01(-1)10(-1)1 \dots$$

It will be useful when it comes to determining the frequency of zeros within certain sequences.

To each  $n \in \mathbb{N}$  we associate the finite sequence  $w_n = \lambda_1 \cdots \lambda_{2^n}$ . By (3.1) the following property of  $w_n$  can be verified.

$$(3.2) \quad w_{n+1}^- = w_n \overline{w_n}.$$

Here the reflection of  $w_n$  w.r.t. the digit set  $\{-1, 0, 1\}$  is defined by  $\overline{w_n} := (-\lambda_1)(-\lambda_2) \cdots (-\lambda_{2^n})$ .

We now prove two lemmas that allow us to prove statements (1) and (2) from Theorem 1.1.

**Lemma 3.1.** *For  $n \geq 2$  the following inequalities hold:*

$$(3.3) \quad \#\{1 \leq i \leq 2^n : \lambda_i = 0\} = 2\#\{1 \leq i \leq 2^{n-1} : \lambda_i = 0\} - 1 \text{ if } n \text{ is even};$$

$$(3.4) \quad \#\{1 \leq i \leq 2^n : \lambda_i = 0\} = 2\#\{1 \leq i \leq 2^{n-1} : \lambda_i = 0\} + 1 \text{ if } n \text{ is odd}.$$

Moreover

$$(3.5) \quad d(w_n) = - \sum_{i=1}^n \left(\frac{-1}{2}\right)^i$$

for all  $n \in \mathbb{N}$ .

*Proof.* We begin by observing that  $w_1 = 10$ , so  $d(w_1) = 1/2$  and (3.5) holds for  $n = 1$ . We now show that (3.3) and (3.4) imply (3.5) via an inductive argument. Let us assume (3.5) is true for odd  $N \in \mathbb{N}$ . Then

$$\begin{aligned} d(w_{N+1}) &= \frac{\#\{1 \leq i \leq 2^{N+1} : \lambda_i = 0\}}{2^{N+1}} \\ &= \frac{2\#\{1 \leq i \leq 2^N : \lambda_i = 0\} - 1}{2^{N+1}} \\ &= d(w_N) - \frac{1}{2^{N+1}} \\ &= -\sum_{i=1}^{N+1} \left(\frac{-1}{2}\right)^i \end{aligned}$$

In our second equality we used (3.3). The case where  $N$  is even is done similarly. Proceeding inductively we may conclude that (3.5) holds assuming (3.3) and (3.4).

It remains to show (3.3) and (3.4) hold. For  $n = 1$  we know that  $w_1 = 10$ , (3.2) therefore implies that the last digit of  $w_2$  equals 1. What is more, repeatedly applying (3.2) we see that the last digit of  $w_n$  equals 0 if  $n$  is odd, and equals 1 if  $n$  is even. Property (3.1) implies that  $\lambda_{2^n+i} = 0$  if  $\lambda_i = 0$  for any  $1 \leq i < 2^n$ . Therefore, when  $n$  is even we see that

$$\begin{aligned} \#\{1 \leq i \leq 2^n : \lambda_i = 0\} &= \#\{1 \leq i \leq 2^{n-1} : \lambda_i = 0\} + \#\{2^{n-1} + 1 \leq i \leq 2^n : \lambda_i = 0\} \\ &= \#\{1 \leq i \leq 2^{n-1} : \lambda_i = 0\} + \#\{1 \leq i \leq 2^{n-1} : \lambda_i = 0\} - 1 \\ &= 2\#\{1 \leq i \leq 2^{n-1} : \lambda_i = 0\} - 1. \end{aligned}$$

Thus (3.3) is proved. Equation (3.4) is proved similarly.  $\square$

Lemma 3.1 determines the frequency of 0's within the finite sequences  $w_n$ . For our proof of Theorem 1.1 we also need to know the frequency of 0's within the sequence  $(\lambda_i)_{i=1}^\infty$ .

**Lemma 3.2.**

$$d((\lambda_i)) = -\sum_{i=1}^{\infty} \left(\frac{-1}{2}\right)^i = \frac{1}{3}.$$

*Proof.* Let us begin by fixing  $\varepsilon > 0$ . Let  $N \in \mathbb{N}$  be sufficiently large such that

$$(3.6) \quad \left| \frac{-\sum_{i=1}^n (-1/2)^i}{1/3} - 1 \right| < \varepsilon$$

for all  $n \geq N$ . Now let us pick  $N' \in \mathbb{N}$  large enough such that

$$(3.7) \quad \frac{\sum_{j=0}^{N-1} 2^j}{N'} < \varepsilon$$



Let  $n \geq N'$  be arbitrary and write  $n = \sum_{j=0}^k \epsilon_j 2^j$ , where we assume  $\epsilon_k = 1$ . By splitting  $(\lambda_i)_{i=1}^n$  into its first  $2^k$  digits, then the next  $2^{k-1}$  digits, then the next  $2^{k-2}$  digits, etc, we obtain:

$$(3.8) \quad \frac{\#\{1 \leq i \leq n : \lambda_i = 0\}}{n} = \frac{\#\{1 \leq i \leq 2^k : \lambda_i = 0\}}{n} + \sum_{l=0}^{k-1} \frac{\#\{\sum_{j=k-l}^k \epsilon_j 2^j + 1 \leq i \leq \sum_{j=k-l-1}^k \epsilon_j 2^j : \lambda_i = 0\}}{n}.$$

By repeatedly applying (3.1) we see

$$(3.9) \quad \begin{aligned} & \#\{1 \leq i \leq \epsilon_{k-l-1} 2^{k-l-1} : \lambda_i = 0\} \\ &= \#\{\epsilon_{k-l} 2^{k-l} + 1 \leq i \leq \epsilon_{k-l} 2^{k-l} + \epsilon_{k-l-1} 2^{k-l-1} : \lambda_i = 0\} \\ &= \dots \\ &= \#\left\{ \sum_{j=k-l}^k \epsilon_j 2^j + 1 \leq i \leq \sum_{j=k-l-1}^k \epsilon_j 2^j : \lambda_i = 0 \right\} \end{aligned}$$

Substituting (3.9) into (3.8) we obtain

$$\begin{aligned} \frac{\#\{1 \leq i \leq n : \lambda_i = 0\}}{n} &= \frac{\#\{1 \leq i \leq 2^k : \lambda_i = 0\}}{n} \\ &+ \sum_{l=0}^{k-1} \frac{\#\{1 \leq i \leq \epsilon_{k-l-1} 2^{k-l-1} : \lambda_i = 0\}}{n}. \end{aligned}$$

By ignoring lower order terms and applying Lemma 3.1, (3.6) and (3.7) we obtain the lower bound

$$\begin{aligned} \frac{\#\{1 \leq i \leq n : \lambda_i = 0\}}{n} &\geq \frac{\#\{1 \leq i \leq 2^k : \lambda_i = 0\}}{n} + \sum_{l=0}^{k-N-1} \frac{\#\{1 \leq i \leq \epsilon_{k-l-1} 2^{k-l-1} : \lambda_i = 0\}}{n} \\ &\geq \frac{(1-\varepsilon)}{3} \left( \frac{2^k}{n} + \sum_{l=0}^{k-N-1} \frac{\epsilon_{k-l-1} 2^{k-l-1}}{n} \right) \\ &= \frac{(1-\varepsilon)}{3} \left( \frac{\sum_{j=0}^k \epsilon_j 2^j - \sum_{j=0}^{N-1} \epsilon_j 2^j}{n} \right) \\ &\geq \frac{(1-\varepsilon)}{3} \left( 1 - \frac{\sum_{j=0}^{N-1} 2^j}{n} \right) \\ &\geq \frac{(1-\varepsilon)^2}{3}. \end{aligned}$$

As  $\varepsilon > 0$  was arbitrary this implies  $\underline{d}((\lambda_i)) \geq 1/3$ . By a similar argument it can be shown that  $\overline{d}((\lambda_i)) \leq 1/3$ . Thus  $d((\lambda_i)) = 1/3$ .  $\square$

Statements (1) and (2) from Theorem 1.1 follow from Lemma 2.3, Lemma 3.1, and Lemma 3.2, when combined with the following results from [15, Lemma 4.12].

**Lemma 3.3.** *Let  $\alpha \in (\alpha_{KL}, 1/2)$ , then there exists  $n^* \in \mathbb{N}$  such that every element of  $\tilde{\mathcal{U}}_\alpha \setminus \{(-1)^\infty, 1^\infty\}$  ends with one of*

$$(0)^\infty, (w_1 \overline{w_1})^\infty, \dots, (w_{n^*} \overline{w_{n^*}})^\infty.$$

**Lemma 3.4.** *Each element of  $\tilde{\mathcal{U}}_{\alpha_{KL}} \setminus \{(-1)^\infty, 1^\infty\}$  is either eventually periodic with period contained in*

$$(0)^\infty, (w_1 \overline{w_1})^\infty, (w_2 \overline{w_2})^\infty, \dots,$$

*or ends with a sequence of the form*

$$(w_0 \overline{w_0})^{k_0} (w_0 \overline{w_{i'_1}})^{k'_0} (w_{i_1} \overline{w_{i_1}})^{k_1} (w_{i_1} \overline{w_{i'_2}})^{k'_1} \dots (w_{i_n} \overline{w_{i_n}})^{k_n} (w_{i_n} \overline{w_{i'_{n+1}}})^{k'_n} \dots,$$

*and its reflection, where  $k_n \geq 0$ ,  $k'_n \in \{0, 1\}$  and*

$$0 < i'_1 \leq i_1 < i'_2 \leq i_2 < \dots \leq i_n < i'_{n+1} \leq i_{n+1} < \dots.$$

By Lemmas 2.3, 3.1 and 3.3 we may conclude

$$D(\alpha) = \left\{0, \frac{\log 2}{-\log \alpha}\right\} \cup \left\{\frac{\log 2}{\log \alpha} \sum_{i=1}^n \left(\frac{-1}{2}\right)^i : 1 \leq n \leq n^*\right\}$$

for some  $n^* \in \mathbb{N}$  for  $\alpha \in (\alpha_{KL}, 1/2)$ . Whilst at the constant  $\alpha_{KL}$  by Lemmas 2.3, 3.1, 3.2 and 3.4 we have

$$D(\alpha_{KL}) = \left\{0, \frac{\log 2}{-\log \alpha_{KL}}, \frac{\log 2}{-3 \log \alpha_{KL}}\right\} \cup \left\{\frac{\log 2}{\log \alpha_{KL}} \sum_{i=1}^n \left(\frac{-1}{2}\right)^i : 1 \leq n < \infty\right\}.$$

Thus statements (1) and (2) from Theorem 1.1 hold. It remains to prove statement (3).

We start by introducing the following finite sequences. Let

$$(3.10) \quad \zeta_n = 0\lambda_1 \dots \lambda_{2^n-1} \text{ and } \eta_n = (-1)\lambda_1 \dots \lambda_{2^n-1}.$$

The following result was proved in [15].

**Lemma 3.5.** *Let  $\alpha \in (1/3, \alpha_{KL})$ , then there exists  $n \in \mathbb{N}$  such that  $\tilde{\mathcal{U}}_\alpha$  contains the subshift of finite type over the alphabet  $\mathcal{A} = \{\zeta_n, \eta_n, \overline{\zeta_n}, \overline{\eta_n}\}$  with transition matrix*

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

*Proof of Theorem 1.1 (3).* Let  $\alpha \in (1/3, \alpha_{KL})$  and let  $n$  be as in Lemma 3.5. So  $\tilde{\mathcal{U}}_\alpha$  contains the subshift of finite type determined by the alphabet  $\mathcal{A}$  and the transition matrix  $A$ . On closer examination we see that this subshift of finite type allows the free concatenation of the words  $\omega_1 = \zeta_n \overline{\zeta_n}$  and  $\omega_2 = \zeta_n \eta_n \overline{\zeta_n}$ . Importantly  $d(\omega_1) < d(\omega_2)$  by (3.10). For any  $c \in [d(\omega_1), d(\omega_2)]$  we can pick a sequence of integers  $k_1, k_2, \dots$  such that the sequence  $(\epsilon_i) = \omega_1^{k_1} \omega_2^{k_2} \omega_1^{k_3} \omega_2^{k_4} \dots$  satisfies  $d((\epsilon_i)) = c$ . Thus by Lemma 2.3 the set  $D(\alpha)$  contains the interval  $[\frac{\log 2}{-\log \alpha} d(\omega_1), \frac{\log 2}{-\log \alpha} d(\omega_2)]$  and our proof is complete.  $\square$

Appealing to standard arguments from multifractal analysis we could in fact show that for any  $c \in (\frac{\log 2}{-\log \alpha} d(\omega_1), \frac{\log 2}{-\log \alpha} d(\omega_2))$  there exists a set of positive Hausdorff dimension within  $\mathcal{U}_\alpha$  with frequency  $c$ .

#### 4. PROOF OF THEOREM 1.2 AND THEOREM 1.3

We start this section by proving Theorem 1.2. Theorem 1.3 will follow almost immediately as a consequence of the arguments used in the proof of Theorem 1.2. To prove Theorem 1.2 we rely on the lexicographic description of  $\tilde{\mathcal{U}}_\alpha$  and  $(\delta_i(\alpha))$  given in Section 2. We take this opportunity to again emphasise that the preliminary results that hold in Section 2 for the alphabet  $\{0, 1, 2\}$  have an obvious analogue that holds for the digit set  $\{-1, 0, 1\}$ .

It is instructive here to state our analogue of the quasi greedy  $\alpha$ -expansion of 1 when  $\alpha = \frac{3-\sqrt{5}}{2}$  and our digit set is  $\{-1, 0, 1\}$ . A straightforward calculation proves that this analogue satisfies

$$(4.1) \quad \left( \delta_i \left( \frac{3-\sqrt{5}}{2} \right) \right) = 1(0)^\infty.$$

We split our proof of Theorem 1.2 into two lemmas.

**Lemma 4.1.** *Let  $\alpha \in (\frac{3-\sqrt{5}}{2}, 1/2)$ , then there exists  $n \in \mathbb{N}$  such that any element of  $\tilde{\mathcal{U}}_\alpha$  cannot contain the sequence  $1(0)^n$  or  $(-1)(0)^n$  infinitely often.*

*Proof.* Suppose  $\alpha \in (\frac{3-\sqrt{5}}{2}, 1/2)$ . Then by Lemma 2.2 and (4.1) we have

$$(4.2) \quad (\delta_i(\alpha)) \prec (1(0)^\infty).$$

For any  $\alpha \in (1/3, 1/2)$  we have  $\delta_1(\alpha) = 1$ . Therefore by (4.2) there exists  $k \geq 0$  such that  $(\delta_i(\alpha))$  begins with the word  $1(0)^k(-1)$ . If a sequence  $(\epsilon_i) \in \tilde{\mathcal{U}}_\alpha$  contained the sequence  $1(0)^{k+1}$  infinitely often, then it is a consequence of Lemma 2.1 for the digit set  $\{-1, 0, 1\}$  that the following lexicographic inequalities would have to hold

$$(4.3) \quad (-1)(0)^k 1 \preceq 1(0)^{k+1} \preceq 1(0)^k(-1).$$

Clearly the right hand side of (4.3) does not hold, therefore  $1(0)^{k+1}$  cannot occur infinitely often. Similarly, one can show that  $(-1)(0)^{k+1}$  cannot occur infinitely often by considering the left hand side of (4.3).  $\square$

**Lemma 4.2.** *If  $\alpha \in (1/3, \frac{3-\sqrt{5}}{2}]$  then for any sequence of natural numbers  $(n_i)$  the sequence*

$$(1(-1))^{n_1} 0^{n_2} (1(-1))^{n_3} 0^{n_4} \dots$$

*is contained in  $\tilde{\mathcal{U}}_\alpha$ .*

*Proof.* Fix a sequence of natural numbers  $(n_i)$ . It is a consequence of Lemma 2.1 and Lemma 2.2 that  $\tilde{\mathcal{U}}_{\frac{3-\sqrt{5}}{2}} \subset \tilde{\mathcal{U}}_\alpha$  for all  $\alpha \in (1/3, \frac{3-\sqrt{5}}{2})$ . Therefore it suffices to show that the sequence

$$(\epsilon_i)_{i=1}^\infty := (1(-1))^{n_1} 0^{n_2} (1(-1))^{n_3} 0^{n_4} \dots$$

is contained in  $\tilde{\mathcal{U}}_{\frac{3-\sqrt{5}}{2}}$ . For all  $n \geq 0$  the following lexicographic inequalities hold

$$(-1)(0)^\infty \prec (\epsilon_i)_{i=n+1}^\infty \prec 1(0)^\infty.$$

Applying Lemma 2.1 we see that  $(\epsilon_i) \in \tilde{\mathcal{U}}_{\frac{3-\sqrt{5}}{2}}$  and our proof is complete.  $\square$

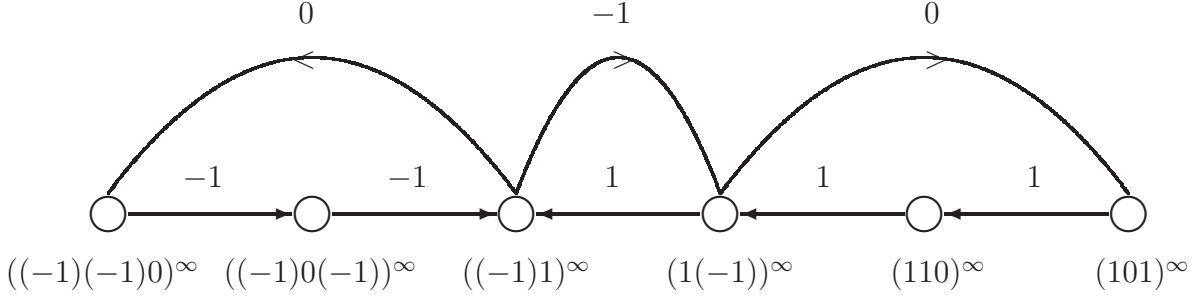
*Proof of Theorem 1.2.* Let  $\alpha \in (\frac{3-\sqrt{5}}{2}, 1/2)$  and let  $N \in \mathbb{N}$  be as in Lemma 4.1. Now let us pick  $a \in (\frac{N}{N+1}, 1)$ . Any  $(t_i) \in \tilde{\mathcal{U}}_\alpha$  with  $\underline{d}((t_i)) = a$  must contain either the sequence  $1(0)^N$  infinitely often or  $(-1)(0)^N$  infinitely often. By Lemma 4.1 this is not possible. Thus by Lemma 2.3 the set  $D_\alpha$  is a proper subset of  $[0, \frac{\log 2}{-\log \alpha}]$  and statement (2) of Theorem 1.2 holds.

By Lemma 2.3 it remains to show that for any  $\alpha \in (1/3, \frac{3-\sqrt{5}}{2}]$  and  $a \in [0, 1]$  there exists  $(t_i) \in \tilde{\mathcal{U}}_\alpha$  such that  $\underline{d}((t_i)) = a$ . The existence of such a  $(t_i)$  now follows from Lemma 4.2 by making an appropriate choice of  $(n_i)$ .  $\square$

We now prove Theorem 1.3. To prove this theorem we require the following technical characterisation of  $S_\alpha$  from [15, Theorem 3.2]. We recall that an infinite sequence  $(\omega_i) \in \{0, 1\}^\mathbb{N}$  is called *strongly eventually periodic* if  $(\omega_i) = IJ^\infty$ , where  $I, J$  are two finite words of the same length and  $I \preceq J$ . Clearly, a periodic sequence is strongly eventually periodic.

**Proposition 4.3.**  *$t \in S_\alpha$  if and only if  $(1 - |t_i|)_{i=1}^\infty$  is strongly eventually periodic.*

*Proof of Theorem 1.3.* Statement (2) of Theorem 1.3 follows from the proof of Theorem 1.2. It is a consequence of our proof that for  $\alpha \in (\frac{3-\sqrt{5}}{2}, 1/2)$  there exists  $\epsilon > 0$  such that  $\underline{d}((t_i)) \notin (1 - \epsilon, 1)$  for all  $(t_i) \in \tilde{\mathcal{U}}_\alpha$ . This statement when combined with Lemma 2.3 implies statement (2) of Theorem 1.3.


 FIGURE 1. A graph generating all  $\alpha$ -expansions of  $t = \sum_{i=1}^{\infty} (-\alpha)^i$ .

To prove statement (1) we remark that for any  $\alpha \in (1/3, \frac{3-\sqrt{5}}{2}]$  and  $n_1, \dots, n_j \in \mathbb{N}$ , the sequence

$$(t_i) = ((1(-1))^{n_1} 0^{n_2} (1(-1))^{n_3} \dots (1(-1))^{n_{j-1}} 0^{n_j})^{\infty}$$

is contained in  $\tilde{\mathcal{U}}_{\alpha}$ . The sequence  $(1 - |t_i|)$  is strongly eventually periodic, therefore by Proposition 4.3 the corresponding  $t$  is contained in  $S_{\alpha}$ . For any  $a \in [0, 1]$  and  $\epsilon > 0$ , we can pick  $n_1, \dots, n_j \in \mathbb{N}$  such that  $|d((t_i)) - a| < \epsilon$ . Applying Lemma 2.3 we may conclude that statement (1) of Theorem 1.3 holds.  $\square$

## 5. EXAMPLES

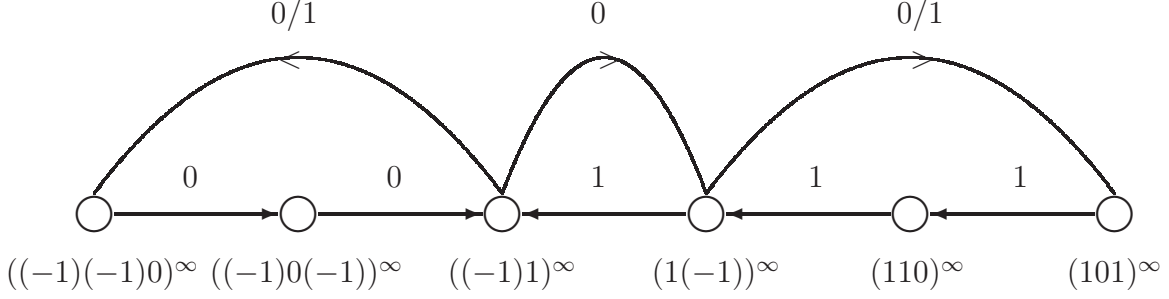
We end our paper with some examples. We start with two examples of an  $\alpha \in (1/3, 1/2)$ , and a  $t \in \Gamma_{\alpha} - \Gamma_{\alpha}$  with a continuum of  $\alpha$ -expansions for which the Hausdorff dimension of  $\Gamma_{\alpha} \cap (\Gamma_{\alpha} + t)$  is explicitly calculable. The approach given in the first example applies more generally to  $\alpha$  the reciprocal of a Pisot number and  $t \in \mathbb{Q}(\alpha)$ . Our second example demonstrates that it is possible for  $t$  to have a continuum of  $\alpha$ -expansions and for  $\Gamma_{\alpha} \cap (\Gamma_{\alpha} + t)$  to be a self-similar set.

**Example 5.1.** Let  $\alpha = 0.449 \dots$  be the unique real root of  $2x^3 + 2x^2 + x - 1 = 0$ . Consider  $t = \sum_{i=1}^{\infty} (-\alpha)^i$ . For this choice of  $\alpha$  the set of  $\alpha$ -expansions of  $t$  is equal to the allowable sequences of edges in Figure 1 that start at the point  $((-1)1)^{\infty}$ .

Using (1.2) we see that  $\Gamma_{\alpha} \cap (\Gamma_{\alpha} + t)$  coincides with those numbers  $\sum_{i=1}^{\infty} \epsilon_i \alpha^i$  where  $(\epsilon_i)$  is a sequence of allowable edges in Figure 2 that start at  $((-1)1)^{\infty}$ .

We let

$$C_n := \left\{ (\epsilon_i)_{i=1}^n \in \{0, 1\}^n : \left[ \sum_{i=1}^n \epsilon_i \alpha^i, \sum_{i=1}^n \epsilon_i \alpha^i + \frac{\alpha^{n+1}}{1 - \alpha} \right] \cap (\Gamma_{\alpha} \cap (\Gamma_{\alpha} + t)) \neq \emptyset \right\}.$$

FIGURE 2. A graph generating  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ 

Given  $\delta_1 \cdots \delta_m \in C_m$  we let

$$C_n(\delta_1 \cdots \delta_m) := \left\{ (\epsilon_i)_{i=1}^{m+n} \in C_{m+n} : (\epsilon_1, \dots, \epsilon_m) = (\delta_1, \dots, \delta_m) \right\}.$$

Making use of standard arguments for transition matrices it can be shown that there exists  $c > 0$  such that

$$(5.1) \quad \frac{\lambda^n}{c} \leq \#C_n \leq c\lambda^n \text{ and } \frac{\lambda^n}{c} \leq \#C_n(\delta_1 \cdots \delta_m) \leq c\lambda^n,$$

for any  $\delta_1 \cdots \delta_m \in C_m$ . Here  $\lambda \approx 1.69562 \dots$  is the unique maximal eigenvalue of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In the following we will show that

$$(5.2) \quad \dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) = \frac{\log \lambda}{-\log \alpha} \approx 0.644297.$$

In fact we show that  $0 < \mathcal{H}^{\frac{\log \lambda}{-\log \alpha}}(\Gamma_\alpha \cap (\Gamma_\alpha + t)) < \infty$ . By (5.1) the upper bound follows from the following straightforward argument:

$$\begin{aligned} \mathcal{H}^{\frac{\log \lambda}{-\log \alpha}}(\Gamma_\alpha \cap (\Gamma_\alpha + t)) &\leq \liminf_{n \rightarrow \infty} \sum_{(\epsilon_i) \in C_n} \text{Diam} \left( \left[ \sum_{i=1}^n \epsilon_i \alpha^i, \sum_{i=1}^n \epsilon_i \alpha^i + \frac{\alpha^{n+1}}{1-\alpha} \right] \right)^{\frac{\log \lambda}{-\log \alpha}} \\ &\leq c\lambda^n \left( \frac{\alpha^{n+1}}{1-\alpha} \right)^{\frac{\log \lambda}{-\log \alpha}} \\ &< \infty \end{aligned}$$

In what follows we use the notation  $\mathcal{I}_n$  to denote the basic intervals corresponding to the elements of  $C_n$ , and  $\mathcal{I}_n(\delta_1 \cdots \delta_m)$  to denote the basic intervals corresponding to elements of  $C_n(\delta_1 \cdots \delta_m)$ .

The proof that  $\mathcal{H}^{-\frac{\log \lambda}{\log \alpha}}(\Gamma_\alpha \cap (\Gamma_\alpha + t)) > 0$  is based upon arguments given in [2] and Example 2.7 from [8]. Let  $\{U_j\}_{j=1}^\infty$  be an arbitrary cover of  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$ . Since  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  is compact we can assume that  $\{U_j\}_{j=1}^p$  is a finite cover. For each  $U_j$  there exists  $l(j) \in \mathbb{N}$  such that  $\frac{\alpha^{l(j)+1}}{1-\alpha} < \text{Diam}(U_j) \leq \frac{\alpha^{l(j)}}{1-\alpha}$ . This implies that  $U_j$  intersects at most two elements of  $\mathcal{I}_{l(j)}$ . This means that for each  $j$  there exists at most two codes  $(\epsilon_1, \dots, \epsilon_{l(j)}), (\epsilon'_1, \dots, \epsilon'_{l(j)}) \in C_{l(j)}$  such that

$$U_j \cap \left[ \sum_{i=1}^{l(j)} \epsilon_i \alpha^i, \sum_{i=1}^{l(j)} \epsilon_i \alpha^i + \frac{\alpha^{l(j)}}{1-\alpha} \right] \neq \emptyset \quad \text{and} \quad U_j \cap \left[ \sum_{i=1}^{l(j)} \epsilon'_i \alpha^i, \sum_{i=1}^{l(j)} \epsilon'_i \alpha^i + \frac{\alpha^{l(j)}}{1-\alpha} \right] \neq \emptyset.$$

Without loss of generality we may assume that  $U_j$  always intersects at least one element of  $\mathcal{I}_{l(j)}$ . Since  $\{U_i\}_{i=1}^p$  is a finite cover there exists  $J \in \mathbb{N}$  such that  $\alpha^J < \text{Diam}(U_i)$  for all  $i$ . By (5.1) the following inequalities hold by counting arguments:

$$\begin{aligned} \frac{\lambda^J}{c} &\leq \#C_J \leq \sum_{j=1}^p \# \left\{ (\epsilon_i) \in C_J : \left[ \sum_{i=1}^J \epsilon_i \alpha^i, \sum_{i=1}^J \epsilon_i \alpha^i + \frac{\alpha^{J+1}}{1-\alpha} \right] \cap U_j \neq \emptyset \right\} \\ &\leq \sum_{j=1}^p \#C_{J-l(j)}(\epsilon_1 \cdots \epsilon_{l(j)}) + \sum_{j=1}^m \#C_{J-l(j)}(\epsilon'_1 \cdots \epsilon'_{l(j)}) \\ &\leq 2c \sum_{j=1}^p \lambda^{J-l(j)} \\ &\leq 2c \sum_{j=1}^p \lambda^J \cdot \alpha^{-l(j) \frac{\log \lambda}{\log \alpha}}. \end{aligned}$$

Cancelling through by  $\lambda^J$  we obtain  $(2c^2)^{-1} \leq \sum_{j=1}^p \alpha^{-l(j) \frac{\log \lambda}{\log \alpha}}$ . Since  $\text{Diam}(U_j)$  is  $\alpha^{l(j)}$  up to a constant term we may deduce that  $\sum_{j=1}^p \text{Diam}(U_j)^{\frac{\log \lambda}{\log \alpha}}$  can be bounded below by a strictly positive constant that does not depend on our choice of cover. This in turn implies  $\mathcal{H}^{-\frac{\log \lambda}{\log \alpha}}(\Gamma_\alpha \cap (\Gamma_\alpha + t)) > 0$ .

By (1.2) we know that

$$(5.3) \quad \dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) \geq \sup_{\tilde{t}} \dim_H \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : \epsilon_i \in \{0, 1\} \cap (\{0, 1\} + \tilde{t}_i) \right\},$$

where the supremum is over all  $\alpha$ -expansions of  $t$ . If  $t$  has countably many  $\alpha$ -expansions, then by the countable stability of the Hausdorff dimension we would have equality in (5.3). In the case where  $t$  has a continuum of  $\alpha$ -expansions it is natural to ask whether equality persists. This example shows that this is not the case. Upon examination of Figure 1 we see that any  $\alpha$ -expansion of  $((-1)1)^\infty$  satisfies  $d((t_i)) \leq 1/3$ . In which case the right hand side of (5.3) can be bounded above by  $\frac{1}{3} \frac{\log 2}{-\log \alpha} \approx 0.281914$ . However by (5.2) this quantity is strictly less than our calculated dimension  $\frac{\log \lambda}{-\log \alpha} \approx 0.644297$ .

**Example 5.2.** Let  $\alpha = \sqrt{2} - 1$  and  $t = \frac{1}{\alpha(\alpha^3-1)} + \frac{1}{\alpha^2(1-\alpha^3)}$ . Then a simple calculation demonstrates that the set of  $\alpha$ -expansions of  $t$  is precisely the set  $\{0(-1)(-1), (-1)10\}^\mathbb{N}$ . Applying (1.2) we see that

$$\Gamma_\alpha \cap (\Gamma_\alpha + t) = \left\{ \sum_{i=1}^{\infty} \epsilon_i \alpha^i : (\epsilon_i) \in \{100, 000, 010, 011\}^\mathbb{N} \right\}$$

This last set is clearly a self-similar set generated by four contracting similitudes of the order  $\alpha^3$ . This self-similar set satisfies the strong separation condition. So

$$\dim_H(\Gamma_\alpha \cap (\Gamma_\alpha + t)) = \frac{\log 4}{-3 \log \alpha}.$$

Each  $\alpha$ -expansion of  $t$  satisfies  $d((t_i)) = 1/3$ . Thus the right hand side of (5.3) can be bounded above by  $\frac{\log 2}{-3 \log \alpha}$ . Thus this choice of  $\alpha$  and  $t$  gives another example where we have strict inequality within (5.3).

We now give an example of an  $\alpha \in (1/3, 1/2)$  and  $t \in \Gamma_\alpha - \Gamma_\alpha$  for which  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  contains only transcendental numbers. For  $\alpha \in (0, 1/3]$  examples are easier to construct, however, when  $\alpha \in (1/3, 1/2)$  the problem of multiple codings arises and a more delicate approach is required. Our examples arise from our proof of Theorem 1.2 and make use of ideas from the well known construction of Liouville.

We call a number  $x \in \mathbb{R}$  a Liouville number if for every  $\delta > 0$  the inequality

$$|x - p/q| \leq q^{-(2+\delta)}$$

has infinitely many solutions. An important result states that every Liouville number is a transcendental number [3]. This result will be critical in what follows.

**Example 5.3.** Let  $p/q \in (1/3, \frac{3-\sqrt{5}}{2})$ . Then there exists  $t \in \mathcal{U}_{p/q}$  such that  $\Gamma_\alpha \cap (\Gamma_\alpha + t)$  only contains Liouville numbers. For any sequences of integers  $(n_k)_{k=1}^\infty$  the sequence

$$(t_i) = (1(-1))^{n_1} 0 (1(-1))^{n_2} 0 \dots$$



is contained in  $\tilde{\mathcal{U}}_{p/q}$ . Now let  $(n_k)$  be a rapidly increasing sequence of integers such that

$$(5.4) \quad \left(\frac{q}{p}\right)^{2n_1+\dots+2n_{k+1}+k+1} \geq q^{k(2n_1+\dots+2n_k+k+3)}$$

Let  $x \in \Gamma_{p/q} \cap (\Gamma_{p/q} + t)$ , then  $x = \sum_{i=1}^{\infty} \epsilon_i \left(\frac{p}{q}\right)^i$  where  $\epsilon_i = 1$  if  $t_i = 1$ ,  $\epsilon_i = 0$  if  $t_i = -1$ , and  $\epsilon_i \in \{0, 1\}$  if  $t_i = 0$ . It follows from our choice of  $(t_i)$  that

$$(\epsilon_i) = (10)^{n_1} \epsilon_{2n_1+1} (10)^{n_2} \epsilon_{2n_1+2n_2+2} \cdots$$

For each  $k \in \mathbb{N}$  we consider the rational

$$(5.5) \quad \frac{p_k}{q_k} := \sum_{i=1}^{2n_1+\dots+2n_k+k} \epsilon_i \left(\frac{p}{q}\right)^i + \left(\frac{p}{q}\right)^{2n_1+\dots+2n_k+k+1} \sum_{i=0}^{\infty} (p/q)^{2i},$$

where  $p_k$  and  $q_k$  are coprime. Either the block 00 or 11 occurs infinitely often within  $(\epsilon_i)$ . So  $p_k/q_k \neq x$ . Importantly, if we expand the right hand side of (5.5) we can bound the denominator by

$$(5.6) \quad q_k \leq q^{2n_1+\dots+2n_k+k+3}.$$

The  $p/q$ -expansion on  $p_k/q_k$  agrees with that of  $x$  upto the first  $(2n_1 + \dots + 2n_{k+1} + k)$  position. Therefore

$$(5.7) \quad |x - p_k/q_k| \leq c \cdot \left(\frac{p}{q}\right)^{2n_1+\dots+2n_{k+1}+k+1}$$

for some constant  $c$ . Combining (5.4), (5.6), and (5.7) we see that for each  $k \in \mathbb{N}$

$$|x - p_k/q_k| \leq c q_k^{-k}.$$

Therefore  $x$  is a Liouville number. Since  $x$  was arbitrary, every  $x \in \Gamma_{p/q} \cap (\Gamma_{p/q} + t)$  is Liouville.

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